The Coin Game Problem

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Python code written by Kailyn Pritchard and Ariel Van Brummelen

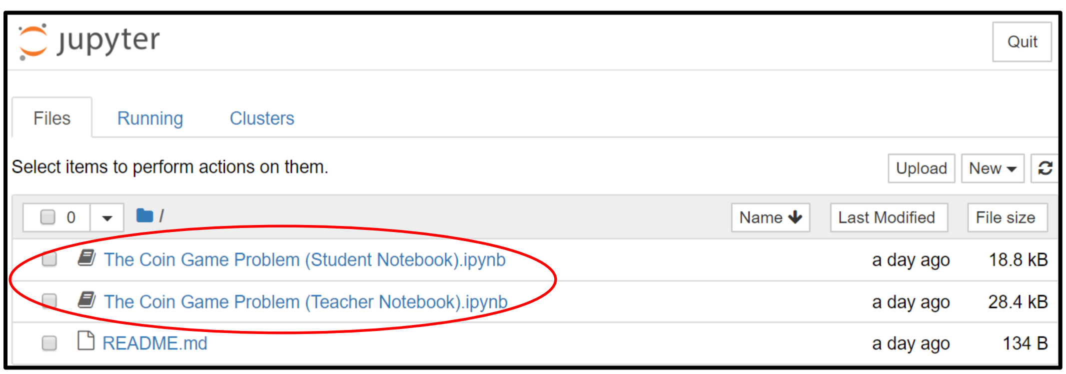
The Coin Game Problem is a free resource for teachers and students, and is part of the Callysto Project (www.callysto.ca), a federally-funded initiative to bring computational thinking and mathematical problem-solving into Grade 5-12 Canadian classrooms.

During the 2018-19 school year, I had the fortune of visiting over a dozen schools and working with 700+ students, sharing rich math problems that incorporated the Callysto technology (a web-based platform known as a Jupyter Notebook, freely accessible to anyone with an Internet connection).

In this Callysto Notebook, we present The Coin Game Problem, a lesson that was first taught to Grade 8 students at Don Ross Middle School in Squamish, and Grade 10 students at Mulgrave School in West Vancouver.

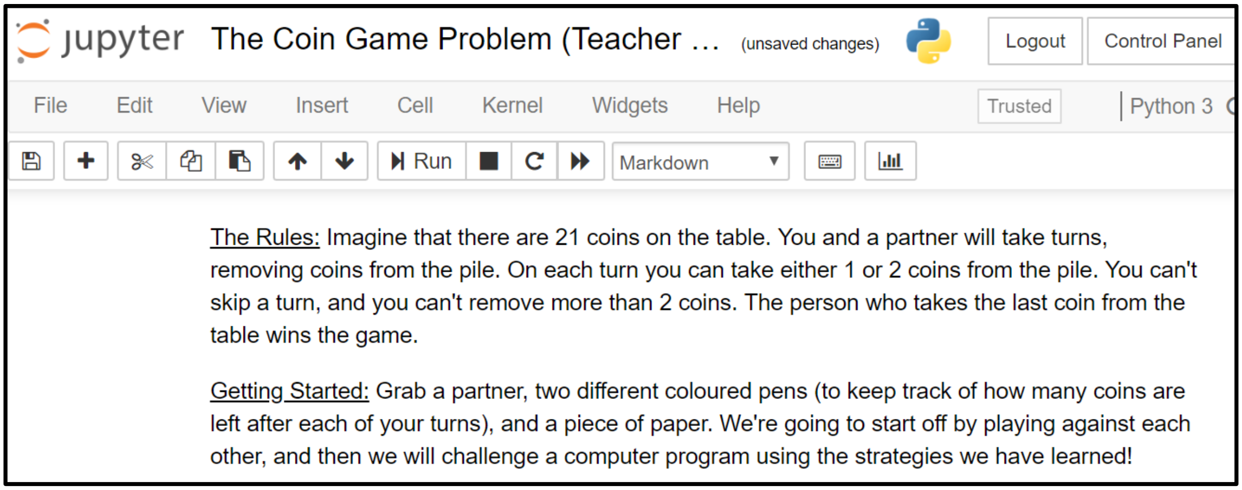
To access this free Notebook, visit www.bit.ly/CallystoCoin.

This link will take you to a screen that looks like this.



Students are to click on the Student Notebook, while this document lists the contents of the Teacher Notebook.

We begin by presenting the following problem, the Coin Game with 21 coins.



The Rules: Imagine that there are 21 coins on the table. You and a partner will take turns, removing coins from the pile. On each turn you can take either 1 or 2 coins from the pile. You can't skip a turn, and you can't remove more than 2 coins. The person who takes the last coin from the table wins the game.

Getting Started: Grab a partner, two different coloured pens (to keep track of how many coins are left after each of your turns), and a piece of paper. We're going to start off by playing against each other, and then we will challenge a computer program using the strategies we have learned!

Have two students come up to the front and demonstrate the game, taking two different coloured pens (say black and blue). Using a red pen, write down the number 21 on the board. The students alternate writing down numbers, subtracting by either one or two each time, until the number 0 is written. And the player who writes 0 wins.

For example, here is one possible play of the game:

21, 19, 18, 16, 14, 13, 11, 10, 8, 6, 4, 3, 2, 0

And in the above example, Player 1 (black) wins.

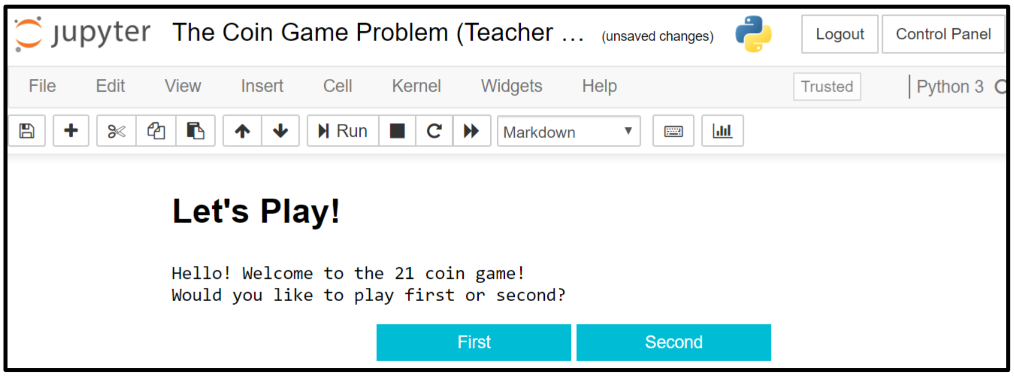
The students are to complete this activity in pairs, either writing down the numbers on a shared sheet of paper, or better yet, standing up and writing down the numbers on a classroom whiteboard. Have them play the game several times, with each student having a chance to play as Player One and as Player Two.

Students will quickly figure out that getting to 3 coins ensures victory. Say Alice and Bob are playing this game, and Alice writes down 3. Since Bob must write down 1 or 2 on his next move, Alice can take what's left and write down 0. In other words, 3 is a "winning position" for the game.

Ask students if there are any other winning positions. Eventually a student will notice that the next winning position is 6. This is because Alice playing 6 implies Bob must play 4 or 5, and then Alice can play 3. And this reduces to the previously-solved winning position, guaranteeing a win for Alice provided she doesn't make a mistake.

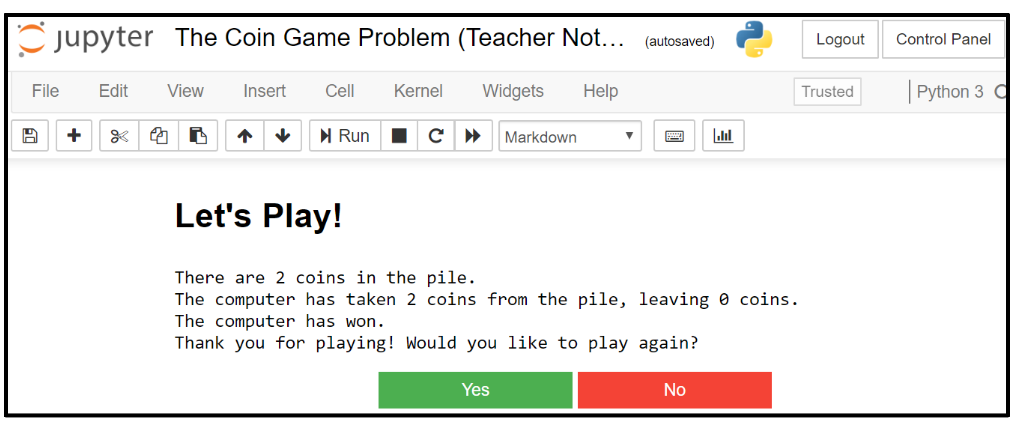
Some students might notice a pattern at this point, and conjecture that the next winning position is 9, since it's also a multiple of 3. And indeed that is true: the winning positions are precisely the multiples of three: 0, 3, 6, 9, 12, etc. The students will soon develop an argument of why that is true.

At this point, the students have developed some key insights, but they haven't yet determined the optimal strategy to play this game. Have them open up a shared computer, load this Notebook, and play the game against the computer. To do so, keep hitting Run until you see the prompt to play this game, just below the "Let's Play!" sign.



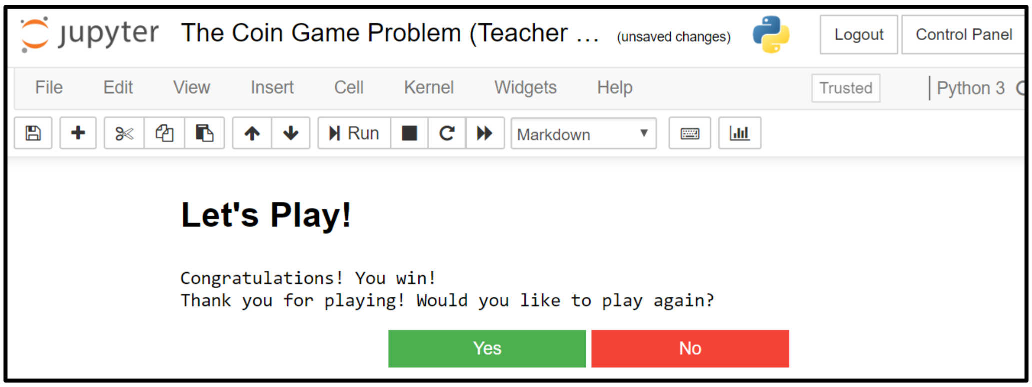
If you'd like to see the Python code that enables this Notebook to function, click on the "Show Code" button at the very top of the Notebook. However, there is no pedagogical requirement to actually show the students the code, other than to mention that the Notebook creators have coded the Coin Game's "optimal strategy", which the students will discover and determine for themselves.

Students have to figure out whether to move First or Second, and what they should do on each move. This will be quite challenging, and inevitably, the students will lose to the computer the first few times they play, since the computer plays optimally, and the computer will win if the students make even one mistake.



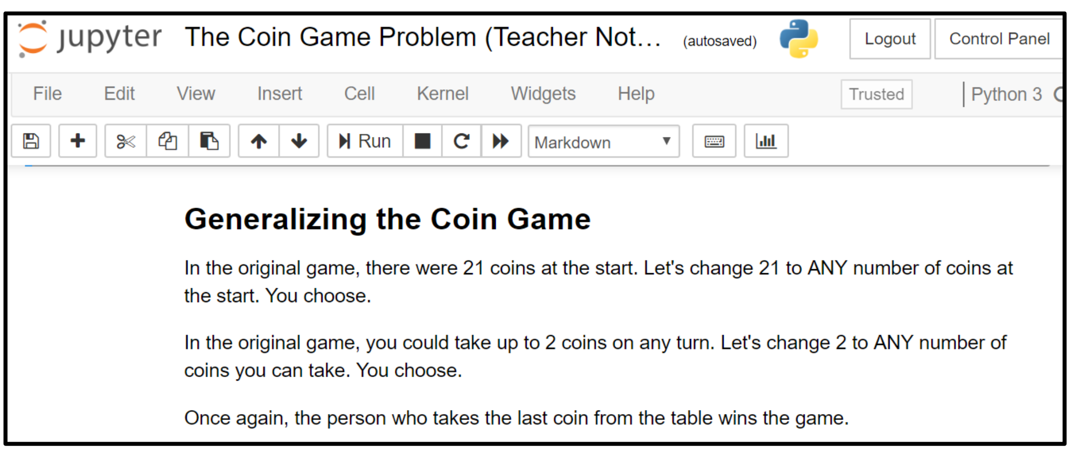
Eventually some pair will determine the correct strategy, and beat the computer. Have these two students share their solution in front of the whole class, showing how they can beat this computer program.

The correct decision is to move Second, and then employ a "reverse-copycat" strategy. If the computer subtracts 1 then you subtract 2, and if the computer subtracts 2 then you subtract 1. This strategy guarantees a win for Player Two.



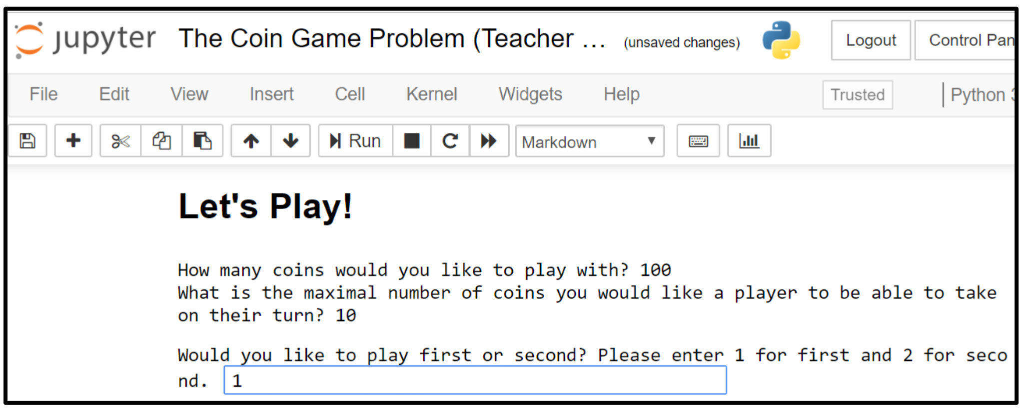
Now have a different group of students explain WHY this strategy works. Here is one possible explanation: since the starting number is a multiple of 3, this reverse-copycat strategy enables Player Two to always ensure that she writes down a multiple of 3: from 21 to 18 to 15 to 12 to 9 to 6 to 3 to 0. Since the winner is the person who writes down 0 (i.e., takes the last coin), Player Two is guaranteed to win this game using this strategy.

Now have the students use this computer to solve the generalized game, changing the starting number of 21 coins to any number N.

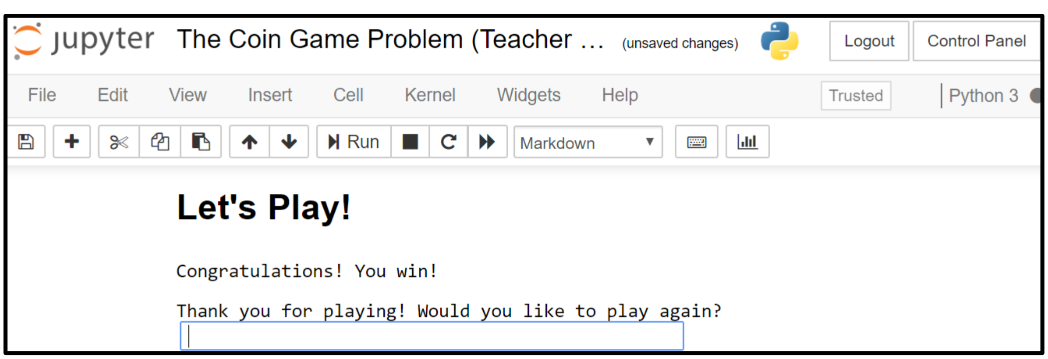


After some analysis, the students will determine the winning strategy for any number of starting coins N. If N is a multiple of 3, they become Player Two and employ the "reverse-copycat" strategy described in the above game with 21 coins. If N is one more than a multiple of 3 (e.g. 4, 7, 10, 13), they become Player One and remove 1 coin on their first move, then follow with the reverse-copycat strategy. And if N is two more than a multiple of 3 (e.g. 5, 8, 11, 14), they become Player One and remove 2 coins on their first move, then follow with the reverse-copycat strategy.

Now have the students generalize the game further, where they can select the starting number of coins N, as well as the maximum number of coins that can be removed on any turn, denoted by the variable M. For example, have the students try the game with N=100 and M=10, and see if they can beat the computer. This game is much harder than the original scenario with N=21 and M=2! The computer plays perfectly, and will win the game if the students make a single mistake.



The solution to the game (N,M) = (100,10) is to be Player One, and take one coin on the first turn. This reduces the game to 99 coins. And then if the computer takes K coins, then we respond by taking 11-K coins. This reduces the number of coins from 99 to 88 to 77 to 66 to 55 to 44 to 33 to 22 to 11 to 0, and we win the game since we get to 0 coins first.



If there is enough class time (or if students wish to analyze this problem outside of class hours), have students work on this generalized problem, and investigate patterns. It turns out that if N is a multiple of M+1, then Player Two has a winning strategy, and if N is not a multiple of M+1, then Player One has a winning strategy. Have students attempt to discover this "theorem" themselves, and determine how they can win against the computer, no matter which values of N and M are chosen.

The Coin Game Problem is an example of a *combinatorial game*, a two-player game that is purely based on skill (rather than luck), whose optimal strategies can be determined through mathematical analysis. Examples of combinatorial games include Tic-Tac-Toe, Checkers, and Chess. There are many applications of combinatorial games to Artificial Intelligence, especially as Game Theory is such an integral part of AI research.

For your strongest students, here is one final challenge problem: suppose that we change the rules of the Coin Game, so that the player who writes down 0 LOSES the game. For this new game, for what values of N and M does Player Two have a winning strategy? Can you prove it, and describe what the winning strategy?

I will purposely not post the solution to that challenge problem here. However, your students are encouraged to write out their solution, scan it as a PDF, and then e-mail it to me (richard.hoshino@gmail.com). I would be happy to write to the student and provide feedback on their solution.

About the Author

Richard Hoshino is a mathematics professor at Quest University Canada, an innovative liberal arts and sciences university located in Squamish, BC. He is the author of "The Math Olympian", has published 30+ research papers across numerous fields, and runs a thriving math consulting business that inspires students, engages teachers, and optimizes businesses. Richard was the 2017 recipient of the Adrien Pouliot Award, awarded by the Canadian Mathematical Society as a lifetime achievement award to celebrate "significant and sustained contributions to mathematics education in Canada". For more information on Richard and his work, please visit www.richardhoshino.com.