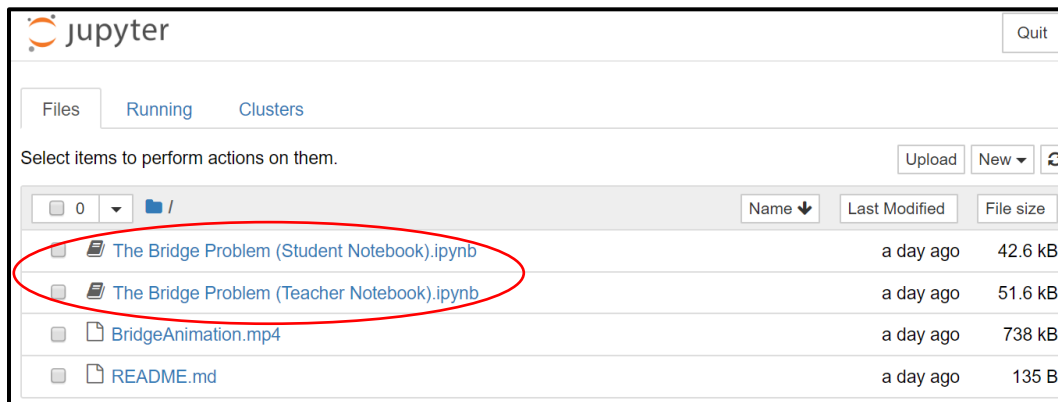# The Bridge Problem

Richard Hoshino

The Bridge Problem is a free resource for teachers and students, and is part of the Callysto Project (www.callysto.ca), a federally-funded initiative to bring computational thinking and mathematical problem-solving into Grade 5-12 Canadian classrooms.

During the 2018-19 school year, I had the fortune of visiting over a dozen schools and working with 700+ students, sharing rich math problems that incorporated the Callysto technology (a web-based platform known as a Jupyter Notebook, freely accessible to anyone with an Internet connection).

In this Callysto Notebook, we present The Bridge Problem, a lesson that was first taught to Grade 8 students at Kitsilano Secondary School in Vancouver, and Grade 9 students at Collingwood School in West Vancouver.

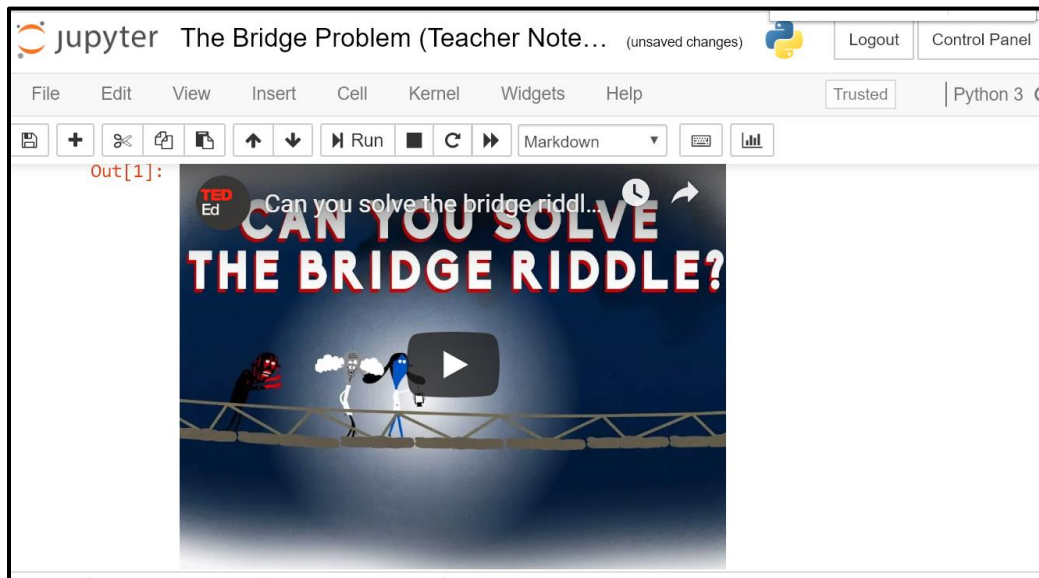To access this free Notebook, visit www.bit.ly/CallystoBridge.

This link will take you to a screen that looks like this.

Students are to click on the Student Notebook, while this document lists the contents of the Teacher Notebook.

We begin by showing the following YouTube video, which presents the Bridge Problem.

https://www.youtube.com/watch?v=7yDmGnA8Hw0

Make sure you cut off the video at the 2:00 mark, just before the solution is revealed.

If your school blocks YouTube, here is the (paraphrased) version of the problem:

Four people (named Alice, Bob, Carol, David) are walking in the dark, and they are sharing one flashlight. They come to an old bridge that can only carry the weight of at most two people at any time.

Alice can cross the bridge in 1 minute, Bob in 2 minutes, Carol in 5 minutes, and David in 10 minutes. Whenever two people walk, they must walk at the pace of the slower person.
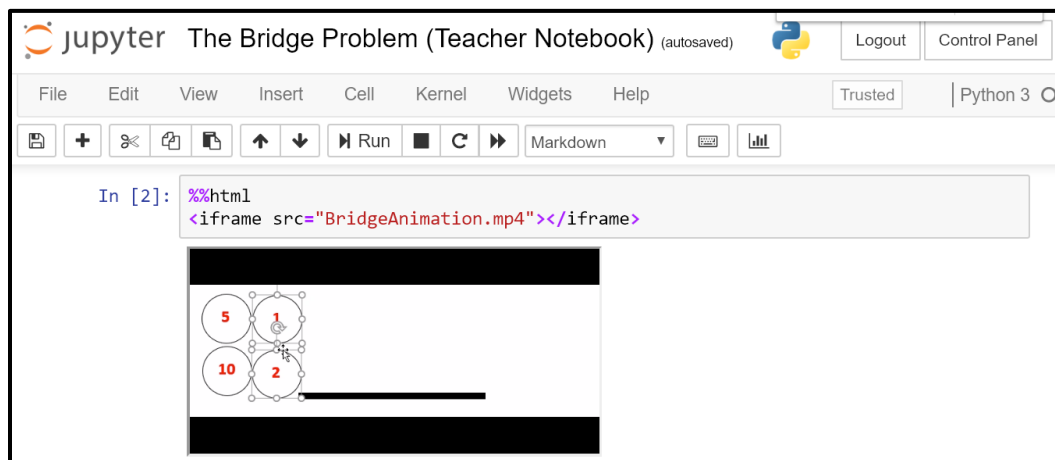
The flashlight must always be with the person (or people) crossing the bridge, in either direction, and cannot be thrown.

The goal is to get all four people across the bridge in 17 minutes.

How can this be done?

Divide students into groups of four, and have them stand and act out this problem, with a pen serving as the lantern (or flashlight). Each student will take on one of the characters, with crossing times A=1, B=2, C=5, and D=10.

While some groups will get into this problem right away, some groups may be stuck and be unsure what is being asked. At that point, have one group demonstrate the "intuitive" solution of sending the fastest person back and forth each time, which leads to a total time of 2+1+5+1+10 = 19 minutes. This is illustrated in the animation provided in the Callysto Notebook.

Students discover that any solution to this problem must involve five steps:

- Step 1: Two people cross the bridge.
- Step 2: One person goes back.
- Step 3: Two people cross the bridge.
- Step 4: One person goes back.
- Step 5: Two people cross the bridge.

Give students enough time to wrestle with the problem, to see if they can get a solution with a total crossing time of 17 minutes. Some groups may give up too soon, believing that the problem is impossible: especially because the sum of the four crossing times, 1+2+5+10, exceeds the target answer of 17.

Students will need to be reminded that when two people cross, they walk at the pace of the slower person, which is the larger number. Run this activity until one (or more) teams gets the correct solution, which is to send 1 and 2 across first, have 1 come back, send 5 and 10 across (!), have 2 come back, and then send 1 and 2 across. This solution requires 2+1+10+2+2 = 17 minutes.
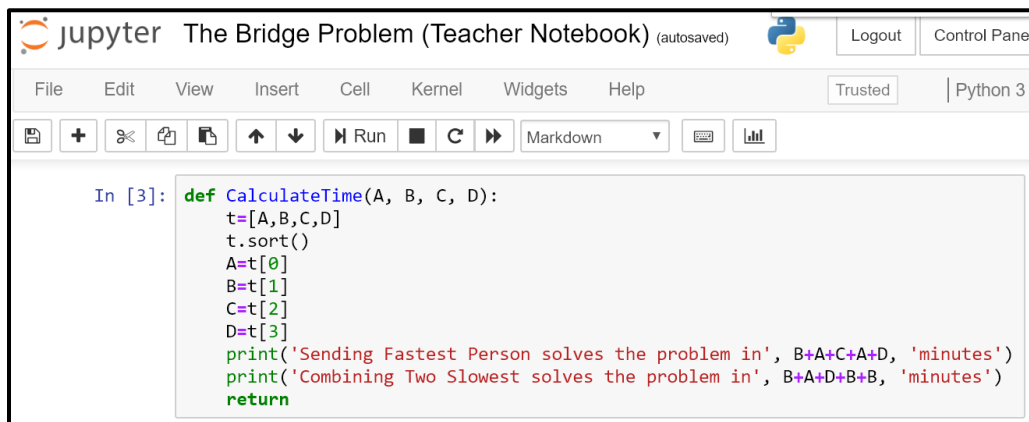
The big insight is that the optimal solution is found by combining the two slowest people, and sending them together. This turns out to be surprisingly faster than the intuitive approach of sending the fastest person back and forth each time. At this point, you may or may not wish to show the remaining part of the YouTube video, to confirm that the video's solution is identical to the students' solution.

Tell the students that you've just discovered two possible "algorithms" to solve this problem:

- Option 1: Always sending the fastest person
- Option 2: Combining the two slowest people

As a class, analyze the two approaches where the crossing times are A, B, C, D, where A<B<C<D. Have them conclude that "always sending the fastest person" has a total time of B+A+C+A+D, and "combining the two slowest people" has a total time of B+A+D+B+B.
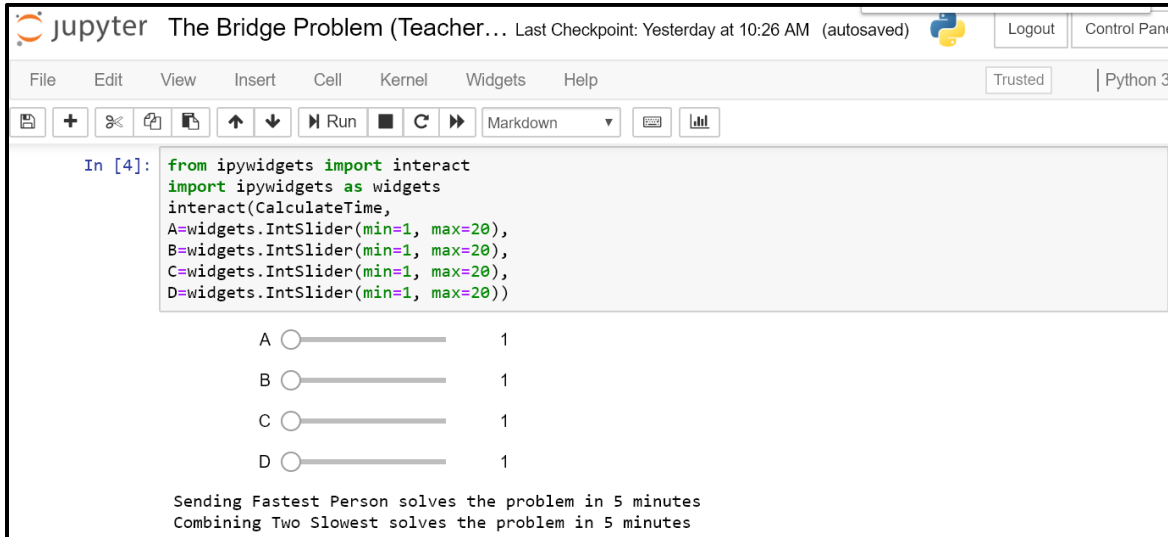
We use the Python programming language to calculate a function called CalculateTime, that outputs the total crossing time given the four inputs A, B, C, D. In a Grade 8 or 9 Math class, we recommend highlighting the two print statements at the end of this code block (and ignoring the rest), while it might be worthwhile to explore this code block with a Programming class, going through the code line by line, to see how the inputs are sorted before the print statements.


```python
In [3]: def CalculateTime(A, B, C, D):
            t=[A,B,C,D]
            t.sort()
            A=t[0]
            B=t[1]
            C=t[2]
            D=t[3]
            print('Sending Fastest Person solves the problem in', B+A+C+A+D, 'minutes')
            print('Combining Two Slowest solves the problem in', B+A+D+B+B, 'minutes')
            return
```

Now have students pair up, and access this Callysto Notebook themselves, using a shared laptop. Have them use the widget slider (see below) to see what happens when they toggle the values of A, B, C, D. They can verify that the values A=1, B=2, C=5, D=10 results in a total crossing time of 19 minutes for the first method and 17 minutes for the second method.

The Python code below creates a "widget slider" using the above CalculateTime function, and instantly outputs the total crossing times for each of our two methods. Once again, we recommend skipping over the details of the code, especially in a Grade 8 or 9 Math Class.

Now ask students to use the widgets to complete the following activity: find values of A, B, C, D, for which we get the SAME total crossing time for the two different methods. For example, the choice A=1, B=3, C=5, D=7 returns a total crossing time of 17 minutes for both methods. Remind the students that A<B<C<D, even though the code automatically sorts the four numbers if they are in the wrong order.

Have the students just focus on the widgets, to determine possible solutions *experimentally*, rather than *algebraically* by finding values for which B+A+C+A+D equals B+A+D+B+B. This part will come later.

After a few minutes, ask each pair to come up to the front and write down the various solutions (A,B,C,D) for which the two methods return the same crossing time. Hopefully the class will have generated at least 20 solutions, such as (1,3,5,7), (4,7,10,18), and (4,7,10,19). Now ask them if they notice anything interesting. They should notice that B always seems to be the average of A and C, and that the variable D is irrelevant.

At this point, tell students that the two methods return the same total time precisely when B+A+C+A+D=B+A+D+B+B. And then this motivates a natural context for teaching students how to simplify equations such as this, by subtracting like terms from both sides. In this particular problem, we have three like terms on both sides (one B, one A, one D), and so the above equation simplifies to A+C=B+B, which implies that B is indeed the average of A and C, and that the variable D indeed is irrelevant.

We very much like teaching this lesson with Grade 8 and 9 students, before they learn how to simplify and solve equations, especially as this lesson motivates this key topic in the high school math curriculum.

For more advanced classes (say a Grade 11 or 12 class), the above problem will be too easy, and students will quickly be able to show that B=(A+C)/2 using some algebra. So the more interesting problem for these students is the following:

How many different ways can the four people cross the bridge?

Have them investigate this in small groups. The solution is below.

The first move, of sending two people across the bridge, can be done in 6 ways (AB, AC, AD, BC, BD, CD), and then the second move can be done in 2 ways (since either individual can return). Since there are now three people on the left side of the bridge, the third move can be made in 3 ways (since exactly one of the three must stay behind). The fourth move can be made in 3 ways (since there are now three people on the right side of the bridge, and exactly one of them must return). Finally, the fifth move can only be made in 1 way, since the remaining two people must walk across the bridge together. Thus, the correct answer is 6x2x3x3x1 = 108.

For your strongest students, here is one final challenge problem:

Let A, B, C, D be any four positive numbers, representing the crossing times of the four people. Among all 108 ways that the four people can all get to the other side, let M be the minimum total time. Prove that this minimum time must be obtained from one of the two solutions described above, i.e., either M=B+A+C+A+D or M=B+A+D+B+B. (In other words, you are to prove that only 2 of the 108 ways can be optimal, for all A,B,C,D.)

I will purposely not post the solution to that challenge problem here. However, your students are encouraged to write out their solution, scan it as a PDF, and then e-mail it to me (richard.hoshino@gmail.com). I would be happy to write to the student and provide feedback on their solution.

About the Author

Richard Hoshino is a mathematics professor at Quest University Canada, an innovative liberal arts and sciences university located in Squamish, BC. He is the author of "The Math Olympian", has published 30+ research papers across numerous fields, and runs a thriving math consulting business that inspires students, engages teachers, and optimizes businesses. Richard was the 2017 recipient of the Adrien Pouliot Award, awarded by the Canadian Mathematical Society as a lifetime achievement award to celebrate "significant and sustained contributions to mathematics education in Canada". For more information on Richard and his work, please visit www.richardhoshino.com.